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Exact Equivalence of the $D = 4$ Gauged Wess-Zumino-Witten Term and the $D = 5$ Yang-Mills Chern-Simons Term

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Abstract

We derive the full Wess-Zumino-Witten term of a gauged chiral lagrangian in $D = 4$ by starting from a pure Yang-Mills theory of gauged quark flavor in a flat, compactified $D = 5$. The theory is compactified such that there exists a B_5 zero mode, and supplemented with quarks that are "chirally delocalized" with q_L (q_R) on the left (right) boundary (brane). The theory then necessarily contains a Chern-Simons term (anomaly flux) to cancel the fermionic anomalies on the boundaries. The constituent quark mass represents chiral symmetry breaking and is a bilocal operator in $D = 5$ of the form: $\overline{q}_L W q_R + h.c$, where W is the Wilson line spanning the bulk, $0 \leq x^5 \leq R$, and is interpreted as a chiral meson field, $W = \exp(2i\tilde{\pi}/f_\pi)$, where $f_\pi \sim 1/R$. The quarks are integrated out, yielding a Dirac determinant which takes the form of a "boundary term" (anomaly flux return), and is equivalent to Bardeen's counterterm that connects consistent and covariant anomalies. The Wess-Zumino-Witten term then emerges straightforwardly, from the Yang-Mills Chern-Simons term, plus boundary term. The method is systematic and allows generalization of the Wess-Zumino-Witten term to theories of extra dimensions, and to express it in alternative and more compact forms. We give a novel form appropriate to the case of (unintegrated) massless fermions.

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I. INTRODUCTION

In this paper we derive the full Wess-Zumino-Witten term [1, 2] of a gauged chiral lagrangian in $D = 4$ by starting from an $SU(N_f)$ Yang-Mills theory of gauged quark flavor in compactified $D = 5$. The Yang-Mills gauge fields propagate in the bulk, with chiral quarks attached to boundaries (branes) located at $x^5 = 0$ and $x^5 = R$. The quarks are chirally delocalized, *i.e.*, their $SU(N_f)$ flavor anomalies are nonzero on their respective boundaries, but would otherwise cancel if the boundaries were merged.

The boundary conditions on the Yang-Mills gauge fields, B_A, are subject to a minimal set of constraints: *(1)* there exists a massless physical B_5 zero mode that can be identified with mesons and, *(2)* there exists a tower of KK-modes of the spin-1, B_μ^n that is sufficiently rich such that independently valued combinations of these fields, exist on the boundary branes, $B_L = B_\mu(x^\mu, 0)$ and $B_R = B_\mu(x^\mu, R)$. With judicious choices of compactification schemes, such as S_1 or "flipped orbifolds," one can imitate the spectrum of QCD, but this need not be specified presently.

Much of what we say will apply to any theory of new physics in extra dimensions that satisfies *(1)* and *(2)* with chiral delocalization. The reason is that the results are largely homological, *i.e.*, they are determined at the boundary of the bulk, as the integrals over the bulk involving the lower KK-modes are mostly exact expressions. In addition, some inexact (bulk integral) components are generated, reflecting new interactions amongst KK-modes that are contained in the Chern-Simons term [3].

With the chiral quarks attached to the boundaries, ψ_L at $x^5 = 0$ and ψ_R at $x^5 = R$ respectively, a "constituent quark mass term" is introduced of the form $m\overline{\psi}_L W \psi_R + h.c.$. Here W is the Wilson line that spans the gap between the boundary branes, and represents the dynamical chiral condensate of the theory. The Wilson line is identified with the chiral field of mesons:

$$W(x^\mu) = P \exp\left(-i \int_0^R dx^5 B_5(x^\mu, x^5)\right) \equiv \exp(2i\tilde{\pi}(x^\mu)/f_\pi) \tag{1}$$

where $\tilde{\pi} = \pi^a \lambda^a/2$ and $f_\pi = 93$ MeV. This is the reason for having a B_5 zero mode, since we desire that the $\tilde{\pi}$ is physical, and not eaten by a KK-mode. In any imitation of QCD chiral dynamics by an extra dimension, chiral symmetry breaking is intrinsically related to the compactification scale, *i.e.*, $f_\pi \sim 1/R$.

The chiral delocalization of the quarks implies the failure of anomaly cancellation on each brane. This mandates a Chern-Simons term spanning the bulk and terminating on the two quark branes. The fermionic anomalies [4, 5, 6] on the boundary branes under $SU(N_f)$ flavor transformations must be cancelled by the anomalies that arise on the boundaries from the Chern-Simons term under the same gauge transformation. This cancellation condition determines the coefficient of the Chern-Simons term. We must use the *consistent anomalies* of the fermions, *i.e.*, the anomalies that come directly from Feynman diagrams (see [3]). We see below that the Chern-Simons anomaly has precisely the same form as the consistent anomaly of the fermion current [6]. We integrate out the quarks in taking the limit of large m. This generates, through the Dirac determinant, a *boundary term*, which is the effective interaction amongst the gauge fields on the boundaries, B_L and B_R. It arises from triangle and box loops, and has a structure identical to that of the counterterm that maps consistent anomalies into covariant anomalies, as introduced by Bardeen ([6], eq.(45)).

Note that, as a metaphor for the structure of the theory, we can view the Chern-Simons term as an "anomaly flux" that runs from one boundary to the other. We can likewise view the boundary terms as an "anomaly return flux." When added together,

$$\tilde{S} = S_{CS} + S_{boundary} \tag{2}$$

we have a total effective action, $\tilde{S}$, that contains no net anomalies and generates the topological physics of the bosons in a fully gauge invariant way. The low energy effective theory, truncated on the B_5 zero mode and fermions, becomes a chiral lagrangian with the symmetry $SU(N_f) \times SU(N_f)_R$. The B_μ modes can play the role of fundamental gauge fields, or as vector and axial vector mesons, coupled to the mesons.

We find in Section III that $\tilde{S}$ resolves into two classes of terms, homological surface terms, and bulk terms. With this construction, we can straightforwardly derive the full Wess-Zumino-Witten (WZW) term. The leading term, $S_{CS0} \sim \int \mathrm{Tr}(\pi d\pi d\pi d\pi d\pi)$, arises immediately from this construction [3], however it is the leading term in an expansion in mesons π. The fully gauged WZW term, S_{WZW}, emerges as the remaining set of exact boundary terms in $\tilde{S}$. In addition we have an interaction term in the bulk, S_{bulk}, which generates new interactions amongst the KK-modes from the Chern-Simons term. These new bulk interactions were previously studied in detail for QED in [3], and the basic procedure developed there is followed here. Our anomaly free action involves cancellation of anomalies

between S_{bulk} and S_{WZW}. A chiral lagrangian in $D = 4$, such as QCD, has no bulk term, and we would simply omit S_{bulk}, leaving an anomalous $S_{CS0} + S_{WZW}$ as the fully gauged Wess-Zumino-Witten term of the theory. Our results confirm the original analysis of Witten, [2], in the finalized form given by Kaymakcalan, Rajeev and Schechter [10], and Manohar and Moore, [11]. We'll maximally conform to the notation of [10] so that our final results are directly comparable to their eq.(4.18).

We emphasize that our procedure is significantly different from the traditional way in which the full WZW term was originally derived. In the standard approach, pioneered by Witten, one *starts with a chiral theory of mesons* in $D = 5$, and incorporates a $D = 5$ chirally invariant pionic interaction, $\mathrm{Tr}(d\pi d\pi d\pi d\pi d\pi)$, with a quantized coefficient. Upon descending to $D = 4$, this yields the $\mathrm{Tr}(\pi d\pi d\pi d\pi d\pi)$ as a boundary term. This is subsequently gauged, *a posteriori*, by performing gauge transformations, observing new terms that are generated by the transformation, and then compensating these with the addition of gauge field and meson interactions. The main difference is that our present procedure begins with a pure Yang-Mills theory and is *a priori* gauge invariant. The mesons are "born" as we descend by compactification and identify a gauge field B_5 with $\tilde{\pi}$. As a result the full WZW term has a larger gauge symmetry involving the "mesons" together with the gauge fields.

We believe that the present analysis amplifies the structure and significance of the full WZW term, and it illustrates technically better ways to manipulate it, and implies that the WZW term is a more general gauge invariant object. To illustrate this, we show in Section IV how to immediately write down the WZW term in the case where the fermions have small masses, and are not integrated out. This is a particularly instructive example as to how the machinery of the full WZW term operates.

II. THE GENERAL SET-UP

A. Fermionic and Gauge Kinetic Terms

Consider a generic $D = 5$, $SU(N)$ Yang-Mills gauge theory compactified on a physical interval $0 \leq x^5 \leq R$. We have vector potentials, $B^a_A(x)$ and coordinates x^A, where $(A = 0, 1, 2, 3, 5)$, the covariant derivative,

$$D_A = \partial_A - iB_A \qquad\qquad B_A = B^a_A T^a, \tag{3}$$

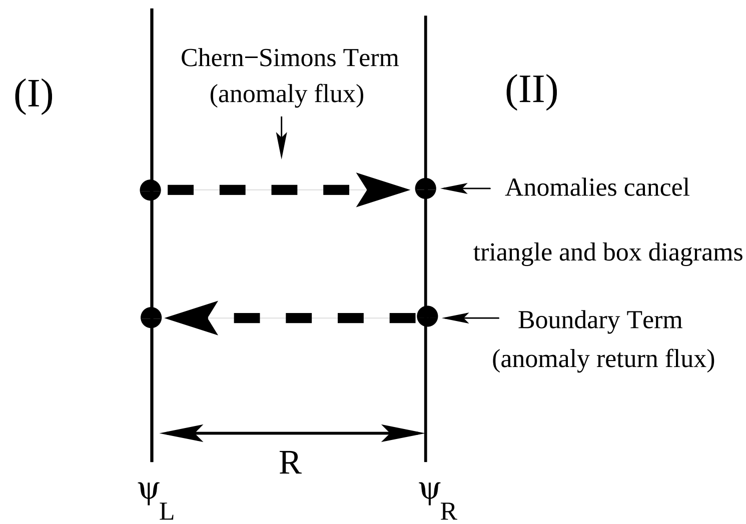

FIG. 1: Orbifold with split, anomalous fermions (quarks). ψ_L (ψ_R) is attached to the $D = 4$ left-boundary brane, I (right-boundary brane, II). Gauge fields propagate in the $D = 5$ bulk, which has a compactification scale R. The fermions have a Wilson line mass term, $m\overline{\psi}_L W \psi_R + h.c.$. The bulk contains a Chern-Simons term. The branes, upon integrating out massive fermions in the large $m >> 1/R$ limit, yield a "boundary term" in the effective action which takes the form of the negative of Bardeen's counterterm. The anomalies from the Chern-Simons term cancel the anomalies from the triangle diagrams on the respective branes so the overall theory is anomaly free. The Chern-Simons action plus boundary term yield the Wess-Zumino-Witten term and a residual bulk interaction amongst KK-modes.

where, *e.g.*, $T^a = \lambda^a/2$ in the adjoint representation of $SU(3)$. The field strength is:

$$G_{AB} = i[D_A, D_B] = \partial_A B_B - \partial_B B_A - i[B_A, B_B]. \tag{4}$$

For completeness, the bulk Yang-Mills kinetic action is:

$$S_0 \;=\; -\frac{1}{2\widetilde{g}^2}\int_0^R dy \int d^4x \,\mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu} - \frac{1}{\widetilde{g}^2}\int_0^R dy \int d^4x \,\mathrm{Tr}\, G_{\mu 5}G^{\mu 5}. \tag{5}$$

where, by examining the zero mode B_μ, the physical coupling constant is $g^2 = \widetilde{g}^2/R$. Beyond this we don't have to be too specific.

We are interested presently in any choice of boundary conditions such that: (1) the lowest B_5 mode is massless and physical, *i.e.*, not eaten by any massive spin-1 modes; (2) we have a tower of KK-modes of B_μ, which may or may not have a massless zero mode, B^0_μ. The low lying KK-modes yield distinct fields $B_L(x^\mu, 0)$ and $B_R(x^\mu, R)$ on the boundaries at $y = 0$ and $y = R$ respectively. There are various choices of boundary conditions for the gauge fields

that can lead to this, and can even imitate QCD. Compactification onto S_1, or "flipped orbifolds" [3], can lead to the physics of interest. We will not discuss these cases presently, as only the most general aspects of the KK-mode spectrum embodied in (1) and (2) are required for this analysis.

We introduce chiral quarks with $N = N_f$ flavors (and N_c ungauged colors) onto the boundaries, I and II, located respectively at $x^5 = 0$ and $x^5 = R$. The fermionic matter action on the boundaries is:

$$S_{kinetic} = \int_I d^4x\, \overline{\psi}_L i\not{D}_L \psi_L + \int_{II} d^4x\, \overline{\psi}_R i\not{D}_R \psi_R \qquad (6)$$

where:

$$D_{L\mu} = \partial_\mu - iB_\mu(x_\mu, 0)\ , \qquad D_{R\mu} = \partial_\mu - iB_\mu(x_\mu, R) \qquad B_\mu = \frac{\lambda^a}{2} B^a_\mu \qquad (7)$$

The B_μ act upon the flavor indices, and:

$$\psi_L = \frac{1-\gamma^5}{2}\psi\ , \qquad \psi_R = \frac{1+\gamma^5}{2}\psi\ . \qquad (8)$$

The ψ_L and ψ_R chiral projections are key ingredients of the theory. This structure can, of course, come about if there is a thin domain wall (kink) at $x^5 = 0$ and an anti-domain wall (anti-kink) at $x^5 = R$, where ψ_L and ψ_R are then the fermionic zero modes. The $B_L = B_\mu(x_\mu, 0)$ ($B_R = B_\mu(x_\mu, R)$) is the left (right) gauge field and is just the bulk gauge field value on the brane at $x^5 = 0$ ($x^5 = R$), the sum over KK-modes on the respective boundary, *e.g.*, $B_R = \sum^n B^n_\mu(x^\mu, R)$.

We further introduce a "constituent quark mass term" of the form:

$$S_{mass} = \int d^4x\, m\overline{\psi}_L W \psi_R + h.c. \qquad W = P\exp\left(-i\int_0^R dx^5 B_5\right) = \exp(2i\tilde{\pi}/f_\pi) \qquad (9)$$

We have defined the Wilson line as the chiral meson field. At this stage the Wilson line contains all the modes of B_5, and it is ambiguous to separate the zero-mode from non-zero modes. This separation is gauge dependent, and we must define the "meson" fields, subsequent to some preparatory manipulations.

B. Transforming to Axial Gauge, $B_5 \to 0$

Consider "gauging away" B_5, which also zeros the Wilson line. We assume that there is no topological invariant associated with the Wilson line, *i.e.*, no nontrivial physical flux

is encircled by the line integral that would be an obstruction to performing this gauge transformation. The transformation removes the Wilson line, hence the "mesons," from the quark mass term.

To implement this, we consider a Wilson line that runs from brane I into the bulk to a position y:

$$V(x^\mu, y) = P\exp\left(-i\int_0^y dx^5\ B_5^0(x^\mu, x^5)\right) \tag{10}$$

and note the essential path-ordering. We now have:

$$i\partial_y V(x^\mu, y) = V(x^\mu, y)B_5^0(x^\mu, y) \qquad i\partial_y V^\dagger(x^\mu, y) = -B_5^0(x^\mu, y)V^\dagger(x^\mu, y) \tag{11}$$

Thus we can consider a gauge transformation:

$$\psi'_L = \psi_L, \qquad \psi'_R = V(R)\psi_R \tag{12}$$

and:

$$\tilde{B}_A(x^\mu, y) = V(B_A + i\partial_A)V^\dagger \tag{13}$$

hence:

$$\tilde{B}_\mu(x^\mu, y) = V(B_\mu + i\partial_\mu)V^\dagger \qquad \tilde{B}_5(x^\mu, y) = V(B_5 + i\partial_y)V^\dagger = 0 \tag{14}$$

The kinetic terms, mass term, and Wilson line thus transform as:

$$\overline{\psi}(i\partial\!\!\!/ + B\!\!\!/\,)\psi = \overline{\psi}'(i\partial\!\!\!/ + \tilde{B}\!\!\!/\,)\psi' \qquad \overline{\psi}_L W\psi_R = \overline{\psi}'_L\psi'_R \qquad W \to V(0)WV^\dagger(R) = 1. \tag{15}$$

The field strengths transform covariantly, $G_{AB} \to VG_{AB}V^\dagger$. Note that fermionic anomalies generated by eq.(12) will be cancelled by the Chern-Simons term.

The physical $\tilde{B}_\mu$ is now a tower of spin-1 fields that have become comingled with the mesons. Hence we wish to extract the $\tilde{\pi}$ meson fields. We now define a particular unitary matrix of the form:

$$\tilde{U}(y) \equiv \exp(2ih(y)\tilde{\pi}/Rf_\pi), \qquad \tilde{\pi} = \pi^a T^a \tag{16}$$

where $y = 0$ ($y = R$) is the left (right) boundary. In this expression $h(y)$ is *ab initio* any monotonic function that takes on values $h(0) = 0$ and $h(R) = 1$. As we'll see momentarily, $h(y)$ controls the longitudinal mixing of $\partial_\mu\tilde{\pi}$ with the pseudovector components of B_μ in $G_{\mu 5}$, and $h(y) = y/R$ is preferred if $\tilde{\pi}$ is a pseudoscalar. We note that $\tilde{U}$ requires no path ordering because $\tilde{\pi}$ is independent of y.

We then define the gauge field $A_\mu(x^\mu, y)$ by the redefinition:

$$\tilde{B}_\mu(x^\mu, y) = \tilde{U}(y)(A_\mu(x^\mu, y) + i\partial_\mu)\tilde{U}^\dagger(y) \tag{17}$$

We emphasize that the definition of eq.(17) is *not a gauge transformation* of the full B_A, but is only a *redefinition* of the $\tilde{B}_\mu$. That is, we do not allow this redefinition to act upon the fermions, so it does not reintroduce the Wilson line into the mass term. The redefinition isolates the mesons from the gauge fields in the first few terms of an expansion in $\tilde{\pi}$, and is sufficient to yield the WZW term below.

As an aside, we note that the redefinition of eq.(17) does not compromise the $D = 4$ gauge invariance of the theory. If we perform a general gauge transformation $\tilde{B}_\mu(x^\mu, y) \to V\tilde{B}_\mu(x^\mu, y)V^\dagger$ we will have an induced gauge transformation $A_\mu \to Y(A_\mu + i\partial_\mu)Y^\dagger$ where $\tilde{U}(y)^\dagger V\tilde{U}(y) = Y$. The generators of Y are $\tilde{U}(y)^\dagger T^a\tilde{U}(y) = T^a_Y$ and satisfy the Lie algebra of $SU(N)$. This redefinition works so long as we don't involve ∂_y in the Y transformation.

The redefinition acts covariantly as a gauge transformation on $G_{\mu\nu}$:

$$G_{\mu\nu}(\tilde{B}) = \tilde{U}(y)G_{\mu\nu}(A)\tilde{U}^\dagger(y) \tag{18}$$

but not so on $G_{\mu 5}$, since we do not transform $\tilde{B}_5 = 0$. We find:

$$\begin{aligned} G_{\mu 5} &= -\partial_y[\tilde{U}(y)(A_\mu + i\partial_\mu)\tilde{U}^\dagger(y)] \\ &= -(2ih'(y)/f_\pi)\tilde{U}(y)(i\partial_\mu\tilde{\pi} + [\tilde{\pi}, A_\mu])\tilde{U}^\dagger(y) - \tilde{U}(y)\partial_y A_\mu\tilde{U}^\dagger(y) \end{aligned} \tag{19}$$

where $\partial_y(\tilde{U}(y)\partial_\mu\tilde{U}^\dagger(y)) = (2ih'(y)/f_\pi)\tilde{U}(y)\partial_\mu\tilde{\pi}\tilde{U}^\dagger(y)$ is exact.

Thus, a $\mathrm{Tr}(G_{\mu 5})^2$ kinetic term contains:

$$-\frac{1}{\tilde{g}^2}\int_0^R dy \int d^4x\, \mathrm{Tr}\, G_{\mu 5}G^{\mu 5} \;=\; \frac{4}{\tilde{g}^2 f_\pi^2}\int_0^R dy \int d^4x\, \mathrm{Tr}\left(h'(y)\partial_\mu\tilde{\pi} - ih'(y)[\tilde{\pi}, A_\mu] - \frac{i}{2}f_\pi\partial_y A_\mu\right)^2 \tag{20}$$

(note a minus sign from raising the 5 index). Setting $A_\mu = 0$ and performing the y integral, (and we assume $h(y)$ is normalized as $\int dy h'(y)^2 = z/R$), we see the emergence of the meson kinetic term, $\mathrm{Tr}(d\tilde{\pi})^2$ with $f_\pi^2 = 4z/R\tilde{g}^2 = 4z/R^2g^2$.

$A_\mu = \sum_n A_\mu^n$, is a tower of KK-modes containing all of the the non-zero-mode longitudinal components of the original B_5, *i.e.*, the longitudinal spin degrees of freedom. These are the Yang-Mills generalizations of "Stueckelberg fields", which have the form, expanding in B_5,

$A^n_\mu \sim B^n_\mu - \partial_\mu B^n_5/M^n + ...$ Under gauge transformations, $\delta B^n_\mu \rightarrow= \partial_\mu \theta^n$ and $\delta B^n_5 = \theta^n M^n$, the A_μ transform covariantly. If the gauge transformation is a valid symmetry of the theory, then we can always bring the massive modes into this covariant form.

The wave-functions of the A^n_μ in y are normalized by the kinetic term, $\int_0^R dy\,\mathrm{Tr}(G^2_{\mu\nu})$. For a typical compactification scheme on the interval $[0, R]$ we may have an A_μ zero mode, with a flat wave-function, $A^0 \sim \sqrt{\tilde{g}^2/R}$ and KK-mode excitations with and $A^n \sim \sqrt{2\tilde{g}^2/R}\cos(n\pi y/R)$, where the $\tilde{g}$ factor rescales the A^n to canonical normalization. Setting $\tilde{\pi} = 0$ in eq.(20) we see that the mass terms for KK-modes are contained in the usual term, $\mathrm{Tr}(\partial_y A_\mu)^2$, and are computed in the mode expansion. With the properly normalized A^n_μ, we have $M^n = \sqrt{2}\pi n/R$.

We also have the longitudinal coupling of the mesons to the vector potentials, contained in the term of the form $\mathrm{Tr}(h'(y)\partial^\mu\tilde{\pi}\partial_y A_\mu)$. This requires a matching of $h(y)$ to the y dependence of the wave-function of the KK-modes in A_μ to establish the parity of the π in a consistent manner. Note that for the typical mode functions, the 1^-, normal parity A^0_μ zero mode (corresponding to the "ρ-meson octet") has $\partial_y A^0_\mu = 0$, thus it decouples from $\partial_\mu\tilde{\pi}$. On the other hand, the first KK-mode, A^1_μ corresponds to an abnormal parity 1^+ state (the "A^1 octet"), and $\partial_y A^1_\mu \sim \sin(\pi y/R)$. By requiring,

$$h'(y) = \frac{1}{R} \qquad\qquad h(y) = \frac{y}{R}. \tag{21}$$

we see that $\partial_\mu\tilde{\pi}$ will be orthogonal to all n-even modes (normal parity) and will couple longitudinally to all n-odd modes with abnormal parity. This fixes $z = 1$, hence $f_\pi = 2/Rg$, and the longitudinal coupling becomes $(gf_\pi/\sqrt{2})\,\mathrm{Tr}(\partial_\mu\tilde{\pi}A^{1\mu})$. We emphasize, however, that the results for the WZW term will be independent of the choice of a particular $h(y)$ provided $h(0) = 0$ and $h(R) = 1$.

An astute reader may be concerned at this point that the $\mathrm{Tr}(G_{\mu 5})^2 \sim \mathrm{Tr}(d\tilde{\pi})^2$ kinetic term has the form of a linear realization of chiral symmetry, and does not represent the nonlinear form embodied in the Wilson line, W. In fact, if we discard the $\mathrm{Tr}(G_{\mu 5})^2$ kinetic term it will not be regenerated by fermion loops, and in this regard can be considered somewhat unnatural. What will be generated by fermion loops is a kinetic term built of $W \equiv U \equiv \exp(2i\tilde{\pi}/f_\pi)$, and an effective redefinition of $G_{\mu 5}$ given by:

$$\begin{aligned} &G_{\mu 5} \rightarrow \tilde{G}_{\mu 5} = \tilde{U}^\dagger[D_\mu, U] \\ &-\frac{1}{\tilde{g}^2}\int_0^R dy \int d^4x\,\mathrm{Tr}\,G_{\mu 5}G^{\mu 5} \rightarrow \frac{1}{g^2}\int d^4x\,\mathrm{Tr}[D_\mu, U][D^\mu, U^\dagger]\,. \end{aligned} \tag{22}$$

where:

$$[D_\mu, U] = \partial_\mu U - iA_L(x^\mu)U + iUA_R(x^\mu) \tag{23}$$

We see that we have now obtained a familiar gauged nonlinear σ-model. This is what we would write directly in a latticized extra dimension [8, 9]. It is an amusing exercise to write $\tilde{G}_{\mu 5}$ in the continuum theory as a power series of operators, containing powers of ∂_y.

An important comment is in order presently, which anticipates the subsequent analysis. The gauge transformation, $V(y)$, and redefinition, $\tilde{U}(y)$, operationally appear to break parity, *i.e.*, the $L \leftrightarrow R$ symmetry of the $D = 5$ theory, since they each start on a preferred brane, I, and run into the bulk toward II. However, this asymmetric choice does not, of course, *physically* break parity, and it has an added bonus: in the following derivation of the WZW term we find that the CS term (the "anomaly flux" term), under $\tilde{U}$, develops a parity asymmetric form (*e.g.*, it will contain terms like $A_R(U^\dagger dU)^3$, where $U = \tilde{U}(R)$, but no corresponding parity conjugates like $A_L(UdU^\dagger)^3$, *etc.*). The boundary term (the "anomaly return flux") will likewise develop a parity asymmetric form, and will contain the parity conjugates, (*e.g.*, the $A_L(UdU^\dagger)^3$ term and no $A_R(U^\dagger dU)^3$). In this way, upon adding the boundary and the CS term, the overall parity symmetry is maintained. The asymmetric choice of $\tilde{U}$ provides a check on the detailed calculation since the parity counterparts are split between the two separate terms, but sum to the parity symmetric final expression. It also reveals the roles of the various components of the WZW term in the masssless fermion case studied in Section IV.

C. The $D = 5$ Yang-Mills Chern-Simons Term

In Appendix A we give some general background information on the $D = 5$ Chern-Simons term. We also show in detail how the consistent anomaly matching to the quarks on the boundary branes leads to quantization of the CS coefficient.

In the original fields, the $D = 5$ CS term takes the form (see Appendix A):

$$S_{CS} \;=\; c\int d^5x\, \epsilon^{ABCDE}\,\mathrm{Tr}\Big(B_A\partial_B B_C\partial_D B_E - \frac{3i}{2}B_A B_B B_C\partial_D B_E - \frac{3}{5}B_A B_B B_C B_D B_E)\Big) \tag{24}$$

where we show that the cancellation of the CS anomalies with the fermion anomalies on the

boundaries requires:

$$c = \frac{N_c}{24\pi^2}. \tag{25}$$

Since we are compactifying x^5, it is important to write the CS term in a form that separates the B_5 and ∂_5 terms. We obtain [8]:

$$S_{CS} = \frac{c}{2}\,\mathrm{Tr}\int d^4x \int_0^R dy\, \Big[(\partial_5 B_\mu)K^\mu + \frac{3}{2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(B_5 G_{\mu\nu}G_{\rho\sigma})\Big], \tag{26}$$

where we find:

$$K^\mu \equiv \epsilon^{\mu\nu\rho\sigma}\left(iB_\nu B_\rho B_\sigma + G_{\nu\rho}B_\sigma + B_\nu G_{\rho\sigma}\right). \tag{27}$$

In deriving this result, some irrelevant total divergences in the $D = 4$ subspace have been discarded.

Now, performing the transformation of eq.(10), leading to axial gauge in which $\tilde{B}_5 = 0$, we see that the Chern-Simons term becomes:

$$S_{CS} \;=\; \frac{c}{2}\epsilon^{\mu\nu\rho\sigma}\int d^4x\int_0^R dy\;\;\mathrm{Tr}\left[\partial_y\tilde{B}_\mu\left(i\tilde{B}_\nu\tilde{B}_\rho\tilde{B}_\sigma + G(\tilde{B})_{\nu\rho}\tilde{B}_\sigma + \tilde{B}_\nu G(\tilde{B})_{\rho\sigma}\right)\right] \tag{28}$$

This is the desired form for S_{CS}.

Form notation will be used throughout the following derivation. This amounts to simply suppressing indices and $\epsilon_{\mu\nu\rho\sigma}$, *e.g.*, rewriting eq.(28) using forms is trivial:

$$S_{CS} \;=\; \frac{c}{2}\,\mathrm{Tr}\int d^4x\int_0^R dy\;(\partial_y\tilde{B})(2d\tilde{B}\tilde{B} + 2\tilde{B}d\tilde{B} - 3i\tilde{B}^3) \tag{29}$$

where the Yang-Mills field strength 2-form is: $G(\tilde{B}) = 2d\tilde{B} - 2i\tilde{B}^2$.

D. The Boundary Term

In the large fermion mass limit, integrating out the fermions, the Dirac determinant yields an effective action, $S_{boundary}$. This is the "boundary term," or "anomaly flux return," and it arises directly from triangle and box loops of the fermions with external gauge fields on the boundaries. We have explicitly computed this in the QED case in [3, 7] where we find that the boundary term is equivalent to the (negative of) Bardeen's counterterm, $\sim (6\pi^2)^{-1}AVdV$. This counterterm adds to the pure fermionic action and transforms the consistent anomalies into covariant ones. We assume that this result generalizes to the case of Yang-Mills and thus postulate the form of the boundary term to be the Yang-Mills generalization. In fact, we

will see that this form of the boundary term is required to maintain the full parity invariance of the resulting WZW term, using the asymmetric $\tilde{U}(y)$.

We expect, therefore, in the large m limit and integrating out the fermions, we will obtain the effective boundary action amongst gauge fields of the form:

$$\begin{aligned} S_{boundary} \; = \; -\frac{c}{2}\int \mathrm{Tr}\,(\frac{1}{2}(G_R\tilde{B}_R+\tilde{B}_R G_R)\tilde{B}_L-\frac{1}{2}(G_L\tilde{B}_L+\tilde{B}_L G_L)\tilde{B}_R \\ +i\tilde{B}_R^3\tilde{B}_L-i\tilde{B}_L^3\tilde{B}_R-\frac{i}{2}(\tilde{B}_R\tilde{B}_L)^2) \end{aligned} \tag{30}$$

where $G_X = G(\tilde{B}_X) = 2d\tilde{B}_X - i\tilde{B}_X^2$. Thus $\tilde{S}$ is the sum of eq.(30) and eq.(28).

III. DERIVATION OF THE WESS-ZUMINO-WITTEN TERM

Our procedure is now to insert the $\tilde{B}$ field, written in terms of A and $\tilde{U}$, into S_{CS} and $S_{boundary}$. We develop the S_{CS} maximally into into exact differentials. The sum $S_{CS} + S_{boundary}$ yields the WZW term in the large m limit.

As a short-hand notation in what follows, the gauge transformed vector potentials can be written as:

$$\tilde{B}_\mu = \tilde{A}_\mu - i\alpha_\mu \tag{31}$$

where:

$$\alpha_\mu = -\tilde{U}\partial_\mu\tilde{U}^\dagger \qquad\qquad \tilde{A}_\mu = \tilde{U}A_\mu\tilde{U}^\dagger \tag{32}$$

where $\tilde{U}(y)$ is defined in eq.(16). Note that both α and $\tilde{A}$ are functions of x_μ and y, and functionals of $\tilde{\pi}$. At the special values of $y=0$ and $y=R$ we have:

$$U \equiv \tilde{U}(R), \qquad \alpha_{L\mu} = 0\ , \qquad \alpha_{R\mu} = \alpha_\mu(x_\mu, R) = -U\partial_\mu U^\dagger\ , \tag{33}$$

and we have:

$$\tilde{A}_{L\mu} = A_{L\mu} = A_\mu(x^\mu, 0)\ , \qquad \tilde{A}_{R\mu} = UA_\mu(x^\mu, R)U^\dagger. \tag{34}$$

Note again the consequence of the parity asymmetry of $\tilde{U}$ which causes $\alpha_{L\mu} = 0$. For future reference we also define the conjugate chiral current:

$$\beta_\mu = U^\dagger\partial_\mu U = U^\dagger\alpha_\mu U \tag{35}$$

Our notation is identical to that of [10] for the purpose of easy comparison (note that the α and β are *not* the usual chiral currents constructed out of ξ, where $\xi^2 = U$, See section IV).

We have the following lemmas:

$$\begin{aligned} d\alpha \ &= \ -dUdU^\dagger = \alpha^2 \qquad\qquad d\beta = -\beta^2 \\ dU \ &= \ -UdU^\dagger U = \alpha U \qquad\qquad dU^\dagger = -U^\dagger dUU^\dagger = -\beta U^\dagger \end{aligned} \tag{36}$$

A. The Chern-Simons Term

We'll consider S_{CS} and $S_{boundary}$ separately. We first substitute $\tilde{B} = \tilde{A} - i\alpha$ into S_{CS} of eq.(29):

$$\begin{aligned} S_{CS} \ = \ & \frac{c}{2}\,\mathrm{Tr}\int d^4x\; dy\; [-i(\partial_y\alpha) + (\partial_y\tilde{A})] \\ & \times(2d\tilde{A}\tilde{A} - 2i\alpha^2\tilde{A} - 2id\tilde{A}\alpha - 4\alpha^3 + 2\tilde{A}d\tilde{A} - 2i\tilde{A}\alpha^2 - 2i\alpha d\tilde{A} \\ & \qquad -3i\tilde{A}^3 - 3\alpha\tilde{A}^2 - 3\tilde{A}\alpha\tilde{A} - 3\tilde{A}^2\alpha + 3i\alpha^2\tilde{A} + 3i\alpha\tilde{A}\alpha + 3i\tilde{A}\alpha^2 + 3\alpha^3) \end{aligned} \tag{37}$$

The trick presently is to write this expression in terms of exact-differentials in y, which leads to exact integrals over y. It is convenient at present to set $R = 1$ and treat y as a dimensionless integration variable running from 0 to 1. We also use "$\int$" to represent "$\int d^4x \ \int_0^1 dy$" in the following, unless otherwise specified. The analysis is straightforward, but a casual reader may skip to the results for S_{CS}, given below in eq.(53). The details of the derivation follow presently.

1. *The Original Wess-Zumino Term:* $(\partial_y\alpha)\alpha^3$

We first isolate the term:

$$S_{CS0} \ = \ i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)\alpha^3 \tag{38}$$

With $\tilde{U} = \exp(2i\tilde{\pi}y/f_\pi)$, we note the *exact* result:

$$\partial_y\alpha = \partial_y\tilde{U}d\tilde{U}^\dagger = \frac{2i}{Rf_\pi}\tilde{U}d\tilde{\pi}\tilde{U}^\dagger \tag{39}$$

If we expand the remaining $\alpha \approx 2iyd\tilde{\pi}/f_\pi$ then we obtain a null result $\propto \mathrm{Tr}(d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi})$, vanishing by cyclicity of the trace. However, to the next order in expansion (consistently for all α factors), we obtain:

$$\alpha \approx \frac{2iy}{f_\pi}d\tilde{\pi} - \frac{2y^2}{f_\pi^2}[\tilde{\pi}, d\tilde{\pi}] + ... \tag{40}$$

Thus, we find, using $c = N_c/24\pi^2$:

$$\begin{aligned} S_{CS0} &= -\frac{2N_c}{3\pi^2 f_\pi^5}\int d^4x\; dy y^4\,\mathrm{Tr}(\tilde\pi d\tilde\pi d\tilde\pi d\tilde\pi d\tilde\pi) + ... \\ &= -\frac{2N_c}{15\pi^2 f_\pi^5}\int d^4x\,\mathrm{Tr}(\tilde\pi d\tilde\pi d\tilde\pi d\tilde\pi d\tilde\pi) + ... \end{aligned} \tag{41}$$

This is the original Wess-Zumino term with Witten's quantized coefficient.

2. The $\alpha^3\tilde A$ Term

We now collect together terms of the form:

$$\begin{aligned} S_{CS\,\alpha^3\tilde A} &= -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(-2id\tilde A\alpha - 2i\alpha d\tilde A - 2i\alpha^2\tilde A - 2i\tilde A\alpha^2 + 3i(\alpha^2\tilde A + \alpha\tilde A\alpha + \tilde A\alpha^2)) \\ &\quad -\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde A)[\alpha^3] \end{aligned} \tag{42}$$

Note that, upon integrating in $D=4$ by parts:

$$\mathrm{Tr}\int(\partial_y\alpha)(d\tilde A\alpha + \alpha d\tilde A) = 2\,\mathrm{Tr}\int(\partial_y\alpha)(\alpha\tilde A\alpha) \tag{43}$$

Thus, we can immediately write:

$$\begin{aligned} S_{CS\,\alpha^3\tilde A} &= -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(i\alpha^2\tilde A + i\tilde A\alpha^2 - i\alpha\tilde A\alpha) - \frac{c}{2}\,\mathrm{Tr}\int d^4x dy(\partial_y\tilde A)[\alpha^3] \\ &= \frac{c}{2}\,\mathrm{Tr}\int d^4x\int_0^1 dy\;\partial_y(\alpha^3\tilde A) \end{aligned} \tag{44}$$

If we now explicitly perform this integral we obtain:

$$S_{CS\,\alpha^3\tilde A} = -\frac{c}{2}\,\mathrm{Tr}(A_R\beta^3) \tag{45}$$

where use has been made $\mathrm{Tr}(\alpha^3\tilde A_R) = \mathrm{Tr}(\alpha^3 U A_R U^\dagger) = \mathrm{Tr}(U^\dagger\alpha^3 U A_R) = \mathrm{Tr}(\beta^3 A_R) = -\mathrm{Tr}(A_R\beta^3)$. We see the operational parity asymmetry of our gauge tranformation leads to the absence of a corresponding parity conjugate term, $-\mathrm{Tr}(A_L\alpha^3)$. As mentioned above, this term will come from the boundary term, and the overall final result will be parity symmetric.

3. The $\alpha\tilde A^3$ Term

We now collect terms of the form:

$$\begin{aligned} S_{CS\,\alpha\tilde A^3} + S^r &= -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(2d\tilde A\tilde A + 2\tilde A d\tilde A - 3i\tilde A^3) \\ &\quad +\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde A)[2d\tilde A\tilde A + 2\tilde A d\tilde A - 3(\alpha\tilde A^2 + \tilde A\alpha\tilde A + \tilde A^2\alpha)] \end{aligned} \tag{46}$$

where S^r is a remainder (see below). We now use $d\tilde{A} = \alpha\tilde{A} + \tilde{A}\alpha + UdAU^\dagger$, and the lemma:

$$\mathrm{Tr}\int \partial_y(\alpha\tilde{A}^3) = \mathrm{Tr}\int[(\partial_y\alpha)(\tilde{A}^3) - (\partial_y\tilde{A})(\tilde{A}^2\alpha - \tilde{A}\alpha\tilde{A} + \alpha\tilde{A}^2)] \tag{47}$$

to write:

$$S_{CS\,\alpha\tilde{A}^3} \;=\; +\frac{c}{2}\,\mathrm{Tr}\int \partial_y(\alpha\tilde{A}^3) \tag{48}$$

and the remainder:

$$S^r \;=\; -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(2d\tilde{A}\tilde{A} + 2\tilde{A}d\tilde{A} - 4i\tilde{A}^3) + \frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde{A})[2\tilde{U}(dAA + AdA)\tilde{U}^\dagger] \tag{49}$$

The remainder is carried into the next set of $\alpha^2\tilde{A}^2$ terms.

4. *The $\alpha^2\tilde{A}^2$ Terms*

Including the remainder from eq.(49), we now have the residual terms:

$$\begin{aligned}
S_{CS\,\alpha^2\tilde{A}^2} \;=\; & -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(2d\tilde{A}\tilde{A} + 2\tilde{A}d\tilde{A} - 4i\tilde{A}^3 - 3\alpha\tilde{A}^2 - 3\tilde{A}\alpha\tilde{A} - 3\tilde{A}^2\alpha) \\
& + \frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde{A})([\tilde{U}(2dAA + 2AdA)\tilde{U}^\dagger] + i\alpha^2\tilde{A} \\
& \quad + 3i\alpha\tilde{A}\alpha + i\tilde{A}\alpha^2 - 2id\tilde{A}\alpha - 2i\alpha d\tilde{A} - 3i\tilde{A}^3)
\end{aligned} \tag{50}$$

Using:

$$\begin{aligned}
-\,\mathrm{Tr}(\partial_y(d\tilde{A}\tilde{A}\alpha) \;&=\; \mathrm{Tr}((\partial_y\alpha)d\tilde{A}\tilde{A}) - \mathrm{Tr}((\partial_y\tilde{A})(d\tilde{A}\alpha - \tilde{A}\alpha^2 + \alpha d\tilde{A})) \\
\mathrm{Tr}(\partial_y(\alpha\tilde{A}d\tilde{A}) \;&=\; \mathrm{Tr}((\partial_y\alpha)\tilde{A}d\tilde{A}) - \mathrm{Tr}((\partial_y\tilde{A})(d\tilde{A}\alpha - \alpha^2\tilde{A} + \alpha d\tilde{A})) \\
\mathrm{Tr}(\partial_y(\alpha\tilde{A}\alpha\tilde{A}) \;&=\; 2\,\mathrm{Tr}((\partial_y\alpha)\tilde{A}\alpha\tilde{A}) - 2\,\mathrm{Tr}((\partial_y\tilde{A})\alpha\tilde{A}\alpha)
\end{aligned} \tag{51}$$

and we therefore have:

$$\begin{aligned}
S_{CS\,\alpha^2\tilde{A}^2} \;=\; & -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(\tilde{U}(3dAA + 3AdA - 4iA^3)\tilde{U}^\dagger) \\
& + \frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde{A})[\tilde{U}(2dAA + 2AdA - 3iA^3)\tilde{U}^\dagger] \\
& + \frac{c}{2}\int \mathrm{Tr}[\partial_y(\tilde{U}dA\tilde{U}^\dagger(-i\tilde{A}\alpha + i\alpha\tilde{A}))] + \frac{ic}{4}\int \mathrm{Tr}[\partial_y(\alpha\tilde{A}\alpha\tilde{A})].
\end{aligned} \tag{52}$$

Note that we have used the identity, $\tilde{U}dA\tilde{U}^\dagger = d\tilde{A} - \alpha\tilde{A} - \tilde{A}\alpha$, to remove the $\tilde{}$ from the $\tilde{A}$ fields that sandwiched between $\tilde{U}$ and $\tilde{U}^\dagger$ in the above expression.

B. Summary of Results for S_{CS}

Collecting the results $S_{CS\ \alpha\tilde{A}^3}$, $S_{CS\ \alpha\tilde{A}^3}$ and $S_{CS\ \alpha^2\tilde{A}^2}$ and performing the exact integrals, we have:

$$
\begin{aligned}
S_{CS} \;=\; & S_{CS0} - \frac{c}{2}\,\mathrm{Tr}(A_R\beta^3) - \frac{c}{2}\,\mathrm{Tr}(A_R^3\beta) - i\frac{c}{4}\,\mathrm{Tr}(A_R\beta A_R\beta) - i\frac{c}{2}\,\mathrm{Tr}[(dA_RA_R + A_RdA_R)\beta] \\
& - i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(\tilde{U}(3dAA + 3AdA - 4iA^3)\tilde{U}^\dagger) \\
& + \frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde{A})[\tilde{U}(2dAA + 2AdA - 3iA^3)\tilde{U}^\dagger]
\end{aligned} \tag{53}
$$

This is the pure Chern-Simons action. As mentioned above, it is parity asymmetric owing to the asymmetric definition of $\tilde{U}(y)$, and the boundary term will restore the parity symmetry.

We now compute the boundary term (anomaly flux return). The final results are quoted in eq.(57).

C. Boundary Term (Anomaly Flux Return)

We substitute eq.(31) into eq.(30) and straightforwardly evaluate. We note that $G_R(\tilde{B}_R) \to UG_R(A_R)U^\dagger$, and $G_L(\tilde{B}_L) \to G_L(A_L)$. The result is:

$$
\begin{aligned}
S_{boundary} \;=\; & \frac{c}{2}\int \mathrm{Tr}\,[(dA_LA_L + A_LdA_L)UA_RU^\dagger - (dA_RA_R + A_RdA_R)U^\dagger A_LU \\
& - i(dA_LA_L + A_LdA_L)\alpha - A_L^3\alpha - A_L\alpha^3 + iA_R^3U^\dagger A_LU - iA_L^3UA_RU^\dagger \\
& - i(dA_RdU^\dagger A_LU - dA_LdUA_RU^\dagger) - (A_RU^\dagger A_LUA_R\beta + A_LUA_RU^\dagger A_L\alpha) \\
& + \frac{i}{2}A_L\alpha A_L\alpha + \frac{i}{2}UA_RU^\dagger A_LUA_RU^\dagger A_L - i(A_LUA_RU^\dagger\alpha^2 - A_RU^\dagger A_LU\beta^2)]
\end{aligned} \tag{54}
$$

where we have used:

$$
-i(A_LUdA_RU^\dagger + U^\dagger dA_RUA_L)\alpha + i(U\beta A_RU^\dagger)\alpha A_L = -i(dA_RdU^\dagger A_LU - dA_LdUA_RU^\dagger) \tag{55}
$$

We see, as in the case of the Chern-Simons anomaly flux, that the result is parity asymmetric, a consequence of our asymmetric choice of $U(y)$. However, we now recover a fully parity symmetric form when we combine the CS term and boundary terms.

D. Full Wess-Zumino-Witten Term

If we now combine all terms, we have the full Wess-Zumino-Witten term, derived from $\tilde{S} = S_{CS} + S_{boundary}$ of eq.(2), and where we now define:

$$\tilde{S} = S_{WZW} + S_{bulk} \tag{56}$$

where:

$$\begin{aligned}
S_{WZW} &= S_{CS0} + \frac{N_c}{48\pi^2}\,\mathrm{Tr}\int d^4x[-(A_L\alpha^3 + A_R\beta^3) - (A_L^3\alpha + A_R^3\beta) \\
&\quad -i((dA_LA_L + A_LdA_L)\alpha + dA_RA_R + A_RdA_R)\beta) + \frac{i}{2}[(A_L\alpha)^2 - (A_R\beta)^2] \\
&\quad -i(A_L^3UA_RU^\dagger - A_R^3U^\dagger A_LU) \\
&\quad +(dA_LA_L + A_LdA_L)UA_RU^\dagger - (dA_RA_R + A_RdA_R)U^\dagger A_LU \\
&\quad -i(dA_RdU^\dagger A_LU - dA_LdUA_RU^\dagger) - (A_LUA_RU^\dagger A_L\alpha + A_RU^\dagger A_LUA_R\beta) \\
&\quad +\frac{i}{2}UA_RU^\dagger A_LUA_RU^\dagger A_L - i(A_LUA_RU^\dagger\alpha^2 - A_RU^\dagger A_LU\beta^2)] \\
\tilde{S}_{CS0} &= -\frac{2N_c}{15\pi^2 f_\pi^5}\int d^4x\,\mathrm{Tr}(\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}) + \ldots
\end{aligned} \tag{57}$$

S_{WZW} is seen to be in complete agreement with Kaymakcalan, Rajeev and Schechter [10] (our result differs by an overall minus sign).

The remaining term, S_{bulk}, is built of inexact integrals over the bulk ($c = N_c/24\pi^2$):

$$\begin{aligned}
S_{bulk} &= -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(\tilde{U}(3dAA + 3AdA - 4iA^3)\tilde{U}^\dagger) \\
&\quad +\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde{A})[\tilde{U}(2dAA + 2AdA - 3iA^3)\tilde{U}^\dagger]
\end{aligned} \tag{58}$$

These are readily developed using the exact results:

$$\partial_y\alpha = \frac{2i}{f_\pi}\tilde{U}(d\tilde{\pi})\tilde{U}^\dagger \qquad\qquad \partial_y\tilde{A} = \partial_y\tilde{U}A\tilde{U}^\dagger = \frac{2i}{f_\pi}\tilde{U}([\tilde{\pi}, A])\tilde{U}^\dagger \tag{59}$$

Note that these expressions are valid to all orders in $\tilde{\pi}$ (not truncated expansions in the $\tilde{\pi}$). Substituting, we see that:

$$S_{bulk} = -\frac{3c}{2f_\pi}\int d^4x\int_0^1 dy\,\mathrm{Tr}(\tilde{\pi}GG) + \frac{c}{2}\int d^4x\int_0^1 dy\,\mathrm{Tr}(\partial_yA)(2dAA + 2AdA - 3iA^3)) \tag{60}$$

We see by comparison to eq.(26) with the matching $A_5 = -2\tilde{\pi}/f_\pi$ that S_{bulk} is just the Chern-Simons term written in the new field variables, A_μ and $\tilde{\pi}$. This reflects bulk interactions

amongst KK-modes. One can obtain the detailed form of these interactions by substituting the wave-functions in the bulk for the KK-modes and performing the dy integrations, as was done previously for QED [3]. The sum of the bulk and boundary contributions make this physics gauge invariant.

Indeed, under a gauge transformation, the WZW term yields the (negative of the) consistent anomaly [10]. Likewise, the bulk term yields the consistent anomaly, and taken together, these contributions cancel. This happens because we have started with a gauge invariant theory. However, a $D = 4$ chiral lagrangian of mesons has no bulk interaction term, and it is anomalous. The result for such a theory is just the S_{WZW} term alone, as is well known.

Eq.(57) is the general result for any $D = 5$ system involving chiral delocalization and bulk Yang-Mills fields. We need only substitute $A_\mu(x, y) = \sum A^n_\mu(x, y)$ and $A_{L\mu}(x) = A_\mu(x, 0)$, and $A_{R\mu}(x) = A_\mu(x, R)$, and identify $2\tilde{\pi}/f_\pi = -\int dyA_5$.

IV. MASSLESS FERMIONS

We can now do something novel with this formalism. It is useful to consider the form of $\tilde{S}$ when the fermions have a small mass and are not integrated out. We envision many possible applications in this limit, since many modern theories are effectively extra dimensional with chiral delocalization. For example, Little Higgs bosons are essentially PNGB's, similar to K-mesons, and the fermion content of these models is effectively a chirally delocalized system in $D = 5$ (usually described by a form of deconstruction). This form of the WZW term would be applicable to Little Higgs interactions with other PNGB's in the theory. For example, we would expect $H + H^\dagger \rightarrow 3\tilde{\pi}$ proceeding though the $\mathrm{Tr}(\pi(d\pi)^4)$ term.

First, it is useful to write the parity asymmetric form, which follows directly from the results derived above. If the fermion mass m is small, and the fermions unintegrated, then the boundary term is not present, but the S_{CS} will be. We can immediately write the form of the effective lagrangian from eq(53):

$$\begin{aligned} S \;=\; & S_{CS0} - \frac{c}{2}\mathrm{Tr}(A_R\beta^3) - \frac{c}{2}\mathrm{Tr}(A_R^3\beta) - i\frac{c}{4}\mathrm{Tr}(A_R\beta A_R\beta) - i\frac{c}{2}\mathrm{Tr}[(dA_RA_R + A_RdA_R)\beta] \\ & + \int_I d^4x\, \overline{\psi}_L(i\partial\!\!\!/ + A\!\!\!/_L)\psi_L + \int_{II} d^4x\, \overline{\psi}_R(i\partial\!\!\!/ + U(A\!\!\!/_R - i\beta)U^\dagger)\psi_R + S_{bulk} \end{aligned} \tag{61}$$

This form is revealing. The theory is fully gauge invariant, and we thus see that a gauge

transformation on A_L commutes with β. Hence, only the S_{bulk} shifts, producing the consistent anomaly that cancels the fermionic anomaly. On the other hand, we can view $U(\not{A}_R - i\beta)U^\dagger$ as a Stueckelberg field, and a shift of $\delta A_R = d\theta_R$ is compensated by $\delta\beta = -d\theta_R$ (recall that $A_5 = -2i\tilde{\pi}/f_\pi$), and no fermionic anomaly is generated. The theory must be invariant under this tranformation, and we see that this happens by a cancellation between S_{bulk} and the first four terms of eq.(66). This provides a shorthand derivation of the fact that, under a gauge transformation, S_{bulk} cancels S_{WZW} in eq.(57).

We can cast the above results into a form that is parity symmetric. We redefine $\tilde{U}(y)$ as ($R = 1$):

$$\tilde{U}(y) = \exp\left(\frac{2i\tilde{\pi}(y-1/2)}{f_\pi}\right) \tag{62}$$

We can define:

$$U(R) = \xi \qquad\qquad U(0) = \xi^\dagger \tag{63}$$

The current $\alpha(y) = -U(y)dU^\dagger(y)$, $\tilde{B} = \tilde{A} - i\alpha$ and $\tilde{A} = U(y)AU^\dagger$ are as defined previously, but now we have:

$$\tilde{B}_L = \xi A_L \xi^\dagger - j_L \qquad\qquad \tilde{B}_R = \xi^\dagger A_L \xi - j_R \tag{64}$$

where:

$$j_L = i\xi d\xi^\dagger \qquad\qquad j_R = -i\xi^\dagger d\xi \tag{65}$$

and S_{CS}, added to the fermionic action, thus becomes, from eq(53):

$$\begin{aligned} S \; = \; & S_{CS0} + S'_{WZW} + S_{bulk} \\ & + \int_I d^4x\, \overline{\psi}_L(i\not{\partial} + \xi\not{A}_L\xi^\dagger - j_L)\psi_L + \int_{II} d^4x\, \overline{\psi}_R(i\not{\partial} + \xi^\dagger\not{A}_R\xi - j_R)\psi_R \end{aligned} \tag{66}$$

where:

$$\begin{aligned} S'_{WZW} \; = \; & -\frac{c}{2}\,\mathrm{Tr}(A_R j_R^3 + A_L j_L^3) - \frac{c}{2}\,\mathrm{Tr}(A_R^3 j_R + A_L^3 j_L) - i\frac{c}{4}\,\mathrm{Tr}(A_R j_R A_R j_R - A_L j_L A_L j_L) \\ & - i\frac{c}{2}\,\mathrm{Tr}[(dA_R A_R + A_R dA_R)j_R + (dA_L A_L + A_L dA_L)j_L] \end{aligned} \tag{67}$$

Note that S_{CS0} and S_{bulk} are unchanged in form. $S'_{WZW} + S_{bulk}$ generates the consistent anomalies to cancel the fermionic anomalies under the various forms of gauge and local chiral tranformations.

V. CONCLUSIONS

We have shown that the Chern-Simons term of a $D = 5$ Yang-Mills theory, together with the boundary terms, yields the full Wess-Zumino-Witten term of a $D = 4$ gauged chiral lagrangian. The present analysis was possible after insights were gleaned from earlier work [3] and [7], which considered in detail the $U(1)$ theory (QED) in $D = 5$ with chiral electrons on boundary branes. In yet another earlier paper we developed the relevant form of the CS-term under compactification of x^5, and we attempted to construct the full WZW term from a pure Yang-Mills theory using latticization (deconstruction) [8]. This approach did not yield the full gauge structure, which we have achieved presently. The present analysis is essentially a detailed application of [3] to Yang-Mills theories.

Let us summarize how the analysis proceeds in general. We begin in a $D = 5$ Yang-Mills theory, compactified in $0 \leq x^5 \leq R$, with chirally delocalized fermions on the boundaries (branes). The theory contains a bulk-filling Chern-Simons term. The chiral fermions have a gauge invariant mass term that is bilocal, $\sim \overline{\psi}_L(x, 0)W\psi_R(x, R) + h.c.$, and involves the Wilson line, $W = P\exp(i\int_0^R B_5 dx^5)$ that spans the bulk. The Wilson line is identified with a chiral field of mesons, $W = \exp(2i\tilde{\pi}/f_\pi)$. A general gauge transformation in the bulk produces anomalies on the boundaries coming from the Chern-Simons term. Likewise, this gauge transformation produces anomalies, coming from the fermions on the boundaries. These anomalies take the consistent form, *i.e.*, they are the direct result of the Feynman triangle loops for the fermions, and have the identical form as the anomalies from the CS term (see Appendix). We demand that these anomalies cancel, and this fixes the coefficient of the CS term, generally to $c = N_c/24\pi^2$.

We now rewrite the CS term into a form that displays separately B_5 and ∂_5. We then perform a master gauge transformation that converts $B_5 \to 0$. This also sets the Wilson line spanning the bulk between the branes to unity. This results in a field $\tilde{B}$ that has the mesons comingled with gauge fields. We thus redefine $\tilde{B} = \tilde{U}A\tilde{U}^\dagger + \alpha$, where $\tilde{U}(y) = \exp(2iy\tilde{\pi}/Rf_\pi)$, and $\alpha = -\tilde{U}d\tilde{U}^\dagger$ is a chiral current built of the mesons. This separates the $\tilde{\pi}$ mesons from the physical gauge fields A. Moreover, the massive components of A are now gauge covariant Stueckelberg fields (see [3]), having "eaten" their longitudinal degrees of freedom contained in the non-zero modes of B_5.

Finally, we integrate out the fermions in the large m limit. This produces effective

interactions (the log of the Dirac determinant) on the boundaries. The form of this effective "Boundary Term" interaction is just Bardeen's counterterm [6] that maps consistent anomalies into covariant ones. We thus have an expression for total action, $\tilde{S}$, the sum of S_{CS}, the Chern-Simons term, and $S_{boundary}$, the boundary terms from the fermionic Dirac determinant. These are functionals of the field $\tilde{B} = \tilde{A} - i\alpha$

We now straightforwardly manipulate the $\tilde{S}$ into terms that are exact forms in the x^5 dimensions, and produce exact integrals, yielding terms that depend only upon the fields on the boundaries. The result is the full Wess-Zumino-Witten term, together with bulk interactions amongst KK-modes mediated by the Chern-Simons term.

We have also given a novel form of the WZW term in the case that the fermions are not integrated out. This reveals the roles of the various components of the full WZW term under the various gauge interactions.

These results apply, in principle, to any theory with chiral delocalization in extra dimensions. If all of the B_5 KK modes are eaten, then we can simply set $\tilde{\pi}$ to zero everywhere in eq.(57). The remaining terms yield the gauge invariant physics of new interactions amongst KK-modes that are generated jointly by the Chern-Simons term and boundary interactions [3].

There are many theories to which these considerations apply, but to which, thus far, this essential physics has not been incorporated. These theories include many incarnations of Randall-Sundrum models, Little Higgs theories, and models of (anomaly) split fermion representations in extra dimensions. The Little Higgs is a PNGB and should participate, like the π or K mesons of QCD, in topological WZW interactions. We further envision applications to string theory, and AdS-CFT QCD as well (for a number of related analyses in the context of SUSY see [15], and a similar approach in M-theory see [14]). The WZW term of gravitation in a split anomaly mode, *e.g.*, in $D = 6$ and $D = 7$, would also be an intriguing application.

A more expansive analysis of the current algebra associated with the $D = 4$ and $D = 5$ chiral/Yang-Mills correspondence is underway, and a number of novel applications is envisioned [16].

Note: It has been brought to my attention that a previous work of Sakai and Sugimoto, carried out in the context of string theory with an ultimately similar configuration to ours, claims to obtain the WZW term from the $D = 5$ CS term [17]. The authors do not discuss

the bulk interaction of our eq.(57), or the unintegrated fermion case of eq.(66). It is unclear as to how the analogue of the boundary term arises in their analysis. Nonetheless, their setup and analysis is quite similar to ours in many respects. It has also been brought to my attention that a little known paper of Novikov [18] first remarked upon the quantization of the coefficient of the Wess-Zumino term, anticipating the classic work of Witten [2].

APPENDIX A: STRUCTURE OF $D = 5$ CHERN-SIMONS TERM

The $D = 5$ Yang-Mills theory of eq.(5), possesses two conserved currents of the form:

$$J_A \; = \; \epsilon_{ABCDE}\,\mathrm{Tr}(G^{BC}G^{DE}), \tag{A1}$$

$$J_A^a \; = \; \epsilon_{ABCDE}\,\mathrm{Tr}\Big(\frac{\lambda^a}{2}\{G^{BC}, G^{DE}\}\Big) \; . \tag{A2}$$

The second current requires that $SU(N)$ possess a d-symbol, hence $N \geq 3$, and it is covariantly conserved, $[D^A,\; J_A^a\;\lambda^a/2] = 0$. These topological currents do not arise from S_0 under local Noetherian variation of the fields.

Why do these currents exist? In fact, these currents describe a special topological soliton in $D = 5$, the "instantonic soliton," that consists of an instanton living on an arbitrary time slice [12]. Owing to eq.(A1) the instantonic soliton carries a conserved charge. Since it is an $SU(2)$ configuration, the current eq.(A2) simply measures how the $SU(2)$ configurations can be imbedded and rotate within the $SU(N)$ group (hence d-symbols measure imbeddings of $SU(2)$ into higher Lie groups). When the theory is compactified according to the rules *(1)* and *(2)*, then this soliton becomes the Skyrmion, eq.(A1) becomes the Goldstone-Wilczek current representing baryon number of the skyrmion, (eq.(A2) becomes a transition flavor current amongst flavors of baryons). The Chern-Simons term when added to the Lagrangian becomes the generator of these currents (see [8]), just as the WZW term is the generator of flavor-skyrmion currents. These correspondences are very tight, even at the level of precise mathematical matchings (*i.e.*, one can infer the form of the full Goldstone-Wilczek current with gauging by matching to eq.(A1) and using a latticized compactification). This correspondence motivates the search for the correspondence between the full WZW term and the Chern-Simons term.

The Chern-Simons term (second Chern character) takes the form:

$$\mathcal{L}_{CS} \; = \; c\epsilon^{ABCDE}\,\mathrm{Tr}\Big(A_A\partial_B A_C\partial_D A_E - \frac{3i}{2}A_A A_B A_C\partial_D A_E - \frac{3}{5}A_A A_B A_C A_D A_E\Big) \tag{A3}$$

and it can be conveniently rewritten as:

$$\mathcal{L}_{CS} \; = \; \frac{c}{4}\epsilon^{ABCDE}\,\mathrm{Tr}\Big(A_A G_{BC} G_{DE} + i A_A A_B A_C G_{DE} - \frac{2}{5} A_A A_B A_C A_D A_E\Big)\;. \tag{A4}$$

It is derived by ascending to $D = 6$ and considering the generalization of the Pontryagin index (a $D = 6$ generalization of the θ-term),

$$\mathcal{L}_P = \epsilon_{ABCDEF}\,\mathrm{Tr}\,G^{AB} G^{CD} G^{EF}. \tag{A5}$$

which can be written as a total divergence,

$$\mathcal{L}_P = -8\partial^F \epsilon_{ABCDEF}\,\mathrm{Tr}\Big(A_A\partial_B A_C \partial_D A_E - \frac{3i}{2} A_A A_B A_C \partial_D A_E - \frac{3}{5} A_A A_B A_C A_D A_E\Big). \tag{A6}$$

Formally, compactifying the sixth dimension and integrating $\mathcal{L}_0$ over the boundary in x^5 leads to $\mathcal{L}_1$. The Chern-Simons term can be constructed in any odd dimension from a general algorithm [13].

Let us perform a generic gauge transformation in the bulk:

$$A_A \to V(A_A + i\partial_A)V^\dagger \qquad \text{where:} \qquad V = \exp(i\theta^a T^a) \tag{A7}$$

and we examine the variation of S_{CS} under this transformation with respect to an infinitesimal $\partial_A\theta^a$. It is most convenient to use eq.(A4), since $G_{AB} \to U^\dagger G_{AB} U$ and we obtain:

$$\begin{aligned} \frac{\delta S_{CS}}{\partial_A \theta^a} \; &= \; c\epsilon^{ABCDE}\,\mathrm{Tr}(T^a \partial_B A_C \partial_D A_E) \\ &\quad -\frac{1}{2} i\,\mathrm{Tr}(T^a A_B A_C(\partial_D A_E) - iT^a A_B(\partial_C A_D)A_E + iT^a(\partial_B A_C)A_D A_E) \end{aligned} \tag{A8}$$

If $D = 5$ is compactified with boundaries located at $x^5 = 0$ and $x^5 = R$, denoted respectively as I and II, then under the gauge transformation we have:

$$\delta S_{CS} \; = \; c\epsilon^{\mu\nu\rho\sigma}\theta^a\,\mathrm{Tr}[T^a(\partial_\mu A_\nu \partial_\rho A_\sigma - \frac{i}{2}(\partial_\mu A_\nu A_\rho A_\sigma - A_\mu \partial_\nu A_\rho A_\sigma + A_\mu A_\nu \partial_\rho A_\sigma)]\Big|_0^R \tag{A9}$$

We refer to this as the “Chern-Simons anomaly.”

We have introduced chiral quarks on the boundaries I and II. The general gauge transformation $U(x^5) = \exp(iT^a\theta^a(x^\mu, x^5)$ acts upon the fermion fields an Wilson line as:

$$\psi_L \to \exp(i\theta(x_\mu, 0))\psi_L\;, \qquad \psi_R \to \exp(i\theta(x_\mu, R))\psi_R \qquad W \to V(0) W V^\dagger(R) \tag{A10}$$

The fermionic action transforms as:

$$S_{branes} \;\rightarrow\; S_{branes} - \int_I d^4x\; \theta^a(x_\mu, 0) Y^a_L - \int_{II} d^4x\; \theta^a(x_\mu, R) Y^a_R \tag{A11}$$

where $Y^a_{L,R}$ is the fermionic anomaly on the corresponding brane. We use Bardeen's result for the *consistent* nonabelian anomalies [6]:

$$\begin{aligned} Y^a_R \;&=\; \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[T^a(\partial_\mu A_{R\nu}\partial_\rho A_{R\sigma} - \frac{i}{2}(\partial_\mu A_{R\nu}A_{R\rho}A_{R\sigma} - A_{R\mu}\partial_\nu A_{R\rho}A_{R\sigma} + A_{R\mu}A_{R\nu}\partial_\rho A_{R\sigma})] \\ Y^a_L \;&=\; -\frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[T^a(\partial_\mu A_{L\nu}\partial_\rho A_{L\sigma} - \frac{i}{2}(\partial_\mu A_{L\nu}A_{L\rho}A_{L\sigma} - A_{R\mu}\partial_\nu A_{L\rho}A_{L\sigma} + A_{L\mu}A_{L\nu}\partial_\rho A_{L\sigma})] \end{aligned} \tag{A12}$$

Note that the consistent anomalies are independent of the mass of the fermion, and they do not decouple in the $m \rightarrow \infty$ limit (while the covariant anomalies do decouple).

Thus we see that the CS anomaly has exactly the same form as the fermionic consistent anomalies. This implies that we can cancel the femionic anomalies against the CS anomalies, if we have:

$$c = \frac{N_c}{24\pi^2}\;. \tag{A13}$$

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